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Pricing Commodity Bonds Using Binomial Option Pricing

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Binomial option pricing offers an easy, flexible, comprehensive method for pricing commodity-linked bonds when there is risk both of default and of changes in commodity prices.

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Commodity-linked bonds have received considerable attention recently as a way to tailor a developing country's debt repayments to its ability to pay. A commodity bond makes repayments subject to fluctuations in the price of the underlying commodity.

Previously, formulas for pricing these bonds were based on the standard continuous-time option-pricing method. Solution of the derived differential equation was difficult even when assumptions were simplified.

Binomial option pricing offers a simpler, more intuitive, and more flexible formula for pricing commodity-linked bonds when there is risk both of default and of changes in commodity prices.

Binomial probability distribution trees are used to develop the pricing formula. Extensions to the model — including the pricing of secondary market debt — are easily incorporated. The technique can also be used to derive, among other things, the implied performance risk of secondary market debt.

This paper is a product of the International Commodity Markets Division, International Economics Department. Copies are available free from the World Bank, 1818 H Street NW, Washington DC 20433. Please contact Julie Raulin, room S7-069, extension 33715.

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I. INTRODUCTION

Interest in commodity-linked securities has increased considerably recently. For the developing countries they offer the possibility of hedging against commodity price risk and thereby enhancing their creditworthiness. Such instruments also link debt repayments to ability to pay (see Priovolos [1987]).

Conventional bonds pay a stated interest rate (coupon) and a fixed principal redeemable at maturity. A commodity bond makes repayments subject to the fluctuations in the price of the underlying commodity. Thus, both the coupon and the principal repayment may be a function of the commodity price. A variety of commodity bond-type instruments can be devised, resulting in different kinds of risk-sharing and return. Two of the more popular variants are the Commodity Convertible Bond (CCB) and the Commodity Linked Bond (CLB). In the CCB the holder can choose on redemption day either the nominal face value or a pre-specified amount of the commodity bundle. The CLB consists of a conventional bond with an attached option or warrant to buy a certain amount of the commodity at a predetermined exercise price. In some markets (not the US) the option can be detached and sold separately. In return for the convertibility/option feature, the issuer receives a lower interest rate.

Issues of commodity bonds can assist liability management by tailoring payments to ability to pay. In a CCB/CLB the coupon provides a "floor" yield. However, when the price of the commodity increases, the yield to maturity for the bond increases and vice versa when the commodity price falls (limited by the floor level).

Formula for pricing commodity-linked bonds have been developed by Schwartz (1982) and Carr (1987). Both use the standard continuous time option pricing method to arrive at a differential equation. The extended form of the

differential equation (incorporating convenience yields) is shown in Appendix A. Schwartz states that the solution to the general problem is difficult even by numerical methods and proceeds to make simplifying assumptions about the nature of the bond to obtain a solution. Even the simplified form of the bond has a mathematically-complex, closed-form solution. The need for a simpler, more intuitive and flexible formulation has been felt.

This note presents a method for pricing commodity-linked bonds in the presence of default risk and commodity price risk. The advantage of this method is that extensions are very simple. Further, the method is more intuitive than the continuous time method while it is equivalent in the limit. Most important, it is flexible and comprehensive. Finally, it can be used to model any bond instrument based on two or more stochastic processes.

Evnine (1983) first extended the Cox, Ross and Rubinstein option pricing model to incorporate an option on two or more stocks. The model developed here is basically a simplification and reformulation of Evnine's model and an application of the model to commodity bonds.

In Part II, a simple version of the bond is priced to make the process transparent. In Part III, the parameters of the model are derived from real world values. In Part IV the model is extended to incorporate the various features that these bonds can include. Part V contains some comparisons of the values obtained by the model with those obtained by Schwartz. Further, some of the additional features are added and priced and observations about some interesting phenomena are made. While Appendix A describes the differential equation that has to be solved and Schwartz's solution to the simplified form, Appendix B shows the logic behind the values of the parameters we have chosen.

II. THE MODEL

Assumptions (all starred terms are values on maturity date)

(i) The commodity-linked bond consists of a zero coupon paying face value F at maturity plus an option to buy a pre-defined quantity of the commodity with value at maturity date equal to P^* at an exercise price of E . The option is European 1/ with maturity date the same as redemption date.

$$B^* = F + \max [0, P^* - E]$$

where B^* is what the bond ought to pay at maturity.

(ii) At maturity, however, the firm's value V^* (consisting of the total value of its assets to its creditors) may be greater than or less than B^* . If the firm is unable to pay, the bondholders get the residual value of the firm. 2/

Therefore, the value of the bond is equal to:

$$\min [V^*, F + \max (0, P^* - E)]$$

(iii) There are no payouts from the firm to the shareholders or bondholders before the maturity date of the bond.

1/ A European option differs from an American one in that it can be exercised only upon expiration rather than at any time.

2/ This is not the case for a sovereign issue. In a developing country when a corporate bondholder defaults governmental authorities often assume foreign obligations.

(iv) The commodity bundle price and the firm value follow multiplicative binomial processes 1/ over discrete periods.

(v) The interest rate is constant and positive.

(vi) The firm's debt consists only of commodity bonds, i.e., there is no senior debt.

(vii) No taxes or transaction costs exist and short sales are allowed. Further, assets are perfectly divisible.

(viii) There is no convenience yield from the commodity.

Assumptions (i), (iii), (v), (vi) and (viii) can be relaxed.

Let the price of the commodity bundle and the value of the firm follow the continuous time diffusion processes described below :

$$\frac{dP}{P} = \mu_p dt + \sigma_p dz_p$$

$$\frac{dV}{V} = \mu_v dt + \sigma_v dz_v$$

$$dz_p dz_v = \sigma_{pv} dt$$

Where σ_p is the volatility of the commodity price, σ_v is the volatility of the firm value and σ_{pv} is the covariance between the two. Also μ_p and μ_v are the drifts of the corresponding price movements.

In our model we will approximate the continuous time diffusion processes with binomial jumps.

1/ See Cox, Ross and Rubinstein (1979).

If the commodity price and firm value moved independently, it would be easy to model the two as a two-step sequence of independent jumps. However, to introduce the covariance term, we need a third step where the price of the commodity bundle and the firm value move together (i.e., as there are two underlying stochastic processes and the processes are not independent, we will assume a three-step process).

Assume three assets; the commodity bundle with price P , the firm with value V and a risk free bond of face value B . Let \hat{r} be $1 +$ the riskless rate of return per period (each jump is considered to occur in a period).

Step 1: Price of commodity bundle P moves up by u_1 with probability q_1 , or down by d_1 with probability $(1 - q_1)$. The value of the firm V accrues at the riskless rate \hat{r} . This is because there is no uncertainty about the value of the firm in this step and hence it is a riskless asset. Therefore it should accrue at the riskless rate.

Step 2: Value of firm moves up by u_2 with probability q_2 or down by d_2 with probability $(1 - q_2)$. The commodity bundle accrues at the riskless rate \hat{r} .

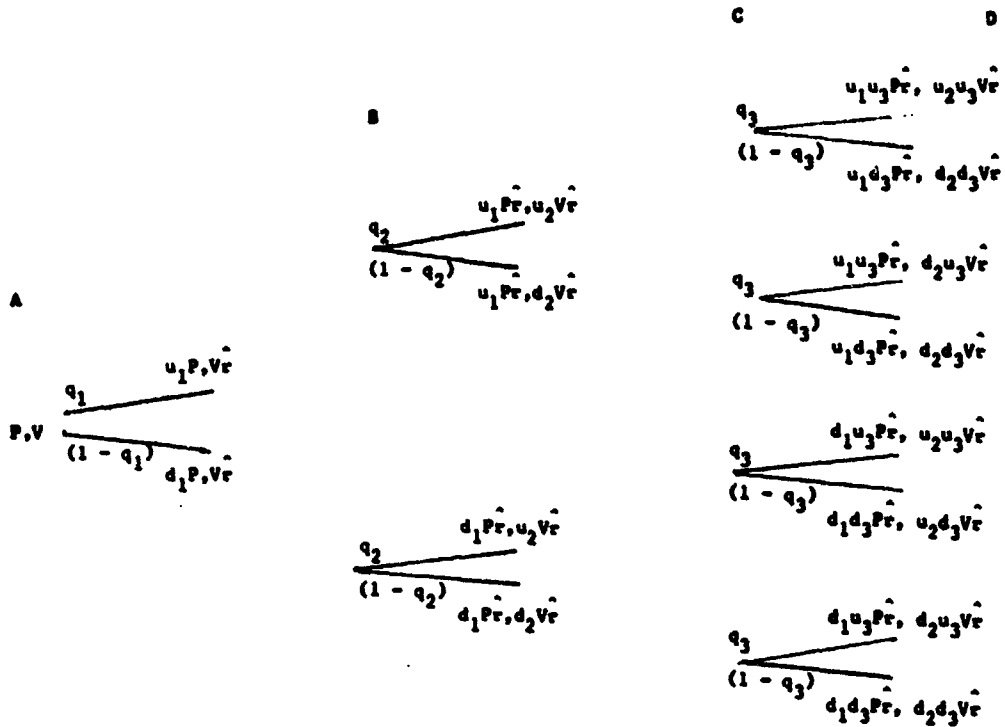
Step 3: P and V together move up by u_3 with probability q_3 or down by d_3 with probability $(1 - q_3)$.

Now folding the tree backwards we can find the expected value of bond at node A (see Figure 1). This would require us to know the probabilities of the upward and downward movement at each step.

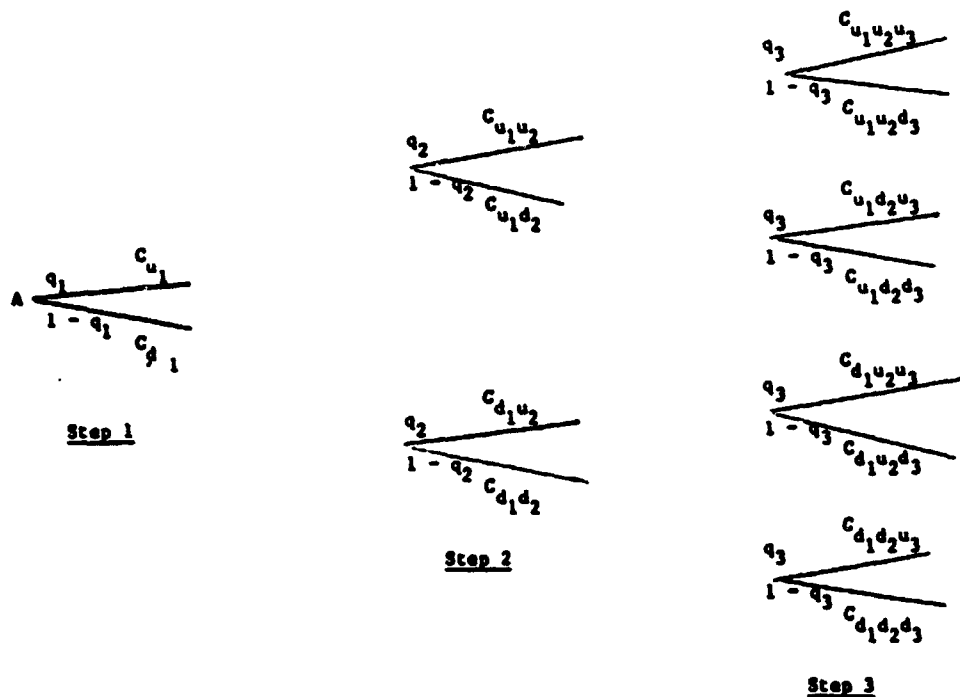
Surprisingly, by creating equivalent portfolios and applying the condition that if two assets have the same value in all possible states of the world in the next period they should have the same value in the current period

Figure 1: The Binomial Tree

At the end of each three-step unit we have the values of the two state variables P and V as follows.



If the bond values at the end of the third step are as shown below,



we find the value at node A of the bond without ever having to know the probability of upward or downward movement.

At node C, let us create a portfolio containing $\Delta \cdot u_1 \hat{r}$ of the commodity bundle and $\Delta \cdot u_2 \hat{r}$ of the firm and B risk free bonds paying \hat{r} per period (we have used B indicating both the risk free bond and the quantity thereof; Δ is some number).

Choose Δ and B such that this portfolio if formed at C has the same value as the commodity bond at D.

i.e., choose Δ and B such that

$\Delta [u_1 \hat{Pr} + u_2 \hat{Vr}] u_3 + \hat{r} B = C_{u_1 u_2 u_3}$ where $C_{u_1 u_2 u_3}$ is the value of the bond after 3 steps, when the price of the commodity bundle has moved up by $u_1 u_3 \hat{r}$ and the value of the firm by $u_2 u_3 \hat{r}$.

Also,

$$\Delta [u_1 \hat{Pr} + u_2 \hat{Vr}] d_3 + \hat{r} B = C_{u_1 u_2 d_3}$$

$$\text{We get } \Delta = \frac{C_{u_1 u_2 u_3} - C_{u_1 u_2 d_3}}{(u_3 - d_3) (u_1 \hat{Pr} + u_2 \hat{Vr})}$$

$$B = \frac{u_3 C_{u_1 u_2 d_3} - d_3 C_{u_1 u_2 u_3}}{(u_3 - d_3) \hat{r}}$$

If there are to be no riskless arbitrage opportunities when the bond in the next period has the same value in all states as the portfolio, we must

have the value of the bond in the present period equal the value of the portfolio in the present period.

$$\begin{aligned}
 C_{u_1 u_2} &= (u_1 \hat{P}r + u_2 V\hat{r}) \Delta + B \\
 &= \left[\left(\frac{\hat{r} - d_3}{u_3 - d_3} \right) C_{u_1 u_2 u_3} + \left(\frac{u_3 - \hat{r}}{u_3 - d_3} \right) C_{u_1 u_2 d_3} \right] / \hat{r} \\
 \text{Setting } P_3 &= \left(\frac{\hat{r} - d_3}{u_3 - d_3} \right) \text{ and } 1 - P_3 = \left(\frac{u_3 - \hat{r}}{u_3 - d_3} \right) \\
 \text{we can write}
 \end{aligned}$$

$$C_{u_1 u_2} = [P_3 C_{u_1 u_2 u_3} + (1 - P_3) C_{u_1 u_2 d_3}] / \hat{r}$$

Similarly all the bond values at nodes below C in Figure 1 can be found in terms of values at the terminal nodes D.

At node B, we can create a portfolio containing Δ_1 of firm value V and B_1 risk free bonds.

Using the same procedure as above, we find

$$\begin{aligned}
 C_{u_1} &= \left[\left(\frac{\hat{r} - d_2}{u_2 - d_2} \right) C_{u_1 u_2} + \left(\frac{u_2 - \hat{r}}{u_2 - d_2} \right) C_{u_1 d_2} \right] / \hat{r} \\
 &= [P_2 C_{u_1 u_2} + (1 - P_2) C_{u_1 d_2}] / \hat{r} \\
 \text{where } P_2 &= \left(\frac{\hat{r} - d_2}{u_2 - d_2} \right)
 \end{aligned}$$

Finally, using a portfolio of Δ_2 of commodity and B_2 of bonds we can show

$$C = [P_1 C_{u_1} + (1 - P_1) C_{d_1}] / \hat{r}$$

$$\text{where } P_1 = \frac{\hat{r} - d_1}{u_1 - d_1}$$

Putting it together we get the recurrence relation for the bond value at period i in terms of the bond values at period $i+3$.

$$\begin{aligned} C = & [P_1 P_2 P_3 C_{u_1 u_2 u_3} + P_1 P_2 (1 - P_3) C_{u_1 u_2 d_3} \\ & + P_1 (1 - P_2) P_3 C_{u_1 d_2 u_3} + P_1 (1 - P_2) (1 - P_3) C_{u_1 d_2 d_3} \\ & + (1 - P_1) P_2 P_3 C_{d_1 u_2 u_3} + (1 - P_1) P_2 (1 - P_3) C_{d_1 u_2 d_3} \\ & + (1 - P_1) (1 - P_2) P_3 C_{d_1 d_2 u_3} + (1 - P_1) (1 - P_2) (1 - P_3) C_{d_1 d_2 d_3}] / \hat{r}^3 \end{aligned}$$

where

$$C_{u_1 u_2 u_3} = \min [u_2 u_3 \hat{r} V, P + \max (u_1 u_3 \hat{r} P - E, 0)]$$

Now let us derive the formula for the bond price after $3n$ periods.

$$\begin{aligned} C = & \hat{r}^{-3n} \left\{ \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^n \left(\frac{n!}{i! (n-i)!} P_1^i (1 - P_1)^{n-i} \right) * \right. \\ & \left. \left(\frac{n!}{j! (n-j)!} P_2^j (1 - P_2)^{n-j} \right) * \right. \end{aligned}$$

$$\left(\frac{n!}{k! (n-k)!} p_3^k (1-p_3)^{n-k} \right) *$$

$$\min [u_2^j u_3^k d_2^n - j d_3^{n-k} \hat{r}^n \quad V, p + \max (0, u_1^i u_3^k d_1^{n-i} d_3^{n-k} \hat{r}^p - E)]$$

III. PARAMETER DETERMINATION

Having derived the recurrence relation for the value of the bond after the 3-step process and thus the bond price after n such 3-step sequences we have to find out how the parameters can be derived from the observed variables.

After $3n$ periods, assuming that there are i steps for process P alone, j steps for process V alone and k steps jointly:

$$P^* = u_1^i \cdot d_1^{n-i} \cdot u_3^k d_3^{n-k} P \hat{r}^n$$

$$\log \left(\frac{P^*}{P} \right) = i \log \frac{u_1}{d_1} + n \log d_1 + k \log \frac{u_3}{d_3} + n \log d_3 + n \log \hat{r}$$

$$E \left[\log \left(\frac{P^*}{P} \right) \right] = E [i] \log \left(\frac{u_1}{d_1} \right) + E [k] \log \left(\frac{u_3}{d_3} \right) + n \log d_1 d_3 \hat{r}$$

$$E \left[\log \left(\frac{P^*}{P} \right) \right] = [q_1 \log \left(\frac{u_1}{d_1} \right) + q_3 \log \left(\frac{u_3}{d_3} \right) + \log d_1 d_3 \hat{r}] n$$

$$\text{var} \left[\log \left(\frac{P^*}{P} \right) \right] = \text{var } i \left[\log \frac{u_1}{d_1} \right]^2 + \text{var } k \left[\log \frac{u_3}{d_3} \right]^2$$

$$+ 2 \text{cov} [i, k] \log \left[\frac{u_1}{d_1} \right] \log \left[\frac{u_3}{d_3} \right]$$

$$= n q_1 (1-q_1) \left[\log \frac{u_1}{d_1} \right]^2 + n q_3 (1-q_3) \left[\log \frac{u_3}{d_3} \right]^2$$

[As covariance $(i, k) = 0$ because the two steps are independent]

Similarly, we can find the mean and the variance for the return on V at the end of the third step by substituting u_2 and d_2 for u_1 and d_1 , and q_2 for q_1 in the above equations.

Finally, to find the covariance term we have

$$\begin{aligned} \text{Covariance} \left(\log\left(\frac{P^*}{P}\right), \log\left(\frac{V^*}{V}\right) \right) &= E \left\{ \log\left(\frac{P^*}{P}\right) \log\left(\frac{V^*}{V}\right) \right. \\ &\left. - E \left[\log\left(\frac{P^*}{P}\right) \right] E \left[\log\left(\frac{V^*}{V}\right) \right] \right\} \end{aligned}$$

After substituting and then taking expectations and some tedious algebra which the reader will be spared we get

$$\begin{aligned} &= [k^2 - (nq_3)^2] \left[\log \left(\frac{u_3}{d_3} \right) \right]^2 \\ &= \text{variance} [k] \left[\log \left(\frac{u_3}{d_3} \right) \right]^2 \\ &= nq_3 (1 - q_3) \left[\log \left(\frac{u_3}{d_3} \right) \right]^2 \end{aligned}$$

For the covariance of the binomial process to equal the covariance of the continuous time process, we have in the limit as $n \rightarrow \infty$

$$\text{cov} \left[\log \left(\frac{P^*}{P} \right), \log \left(\frac{V^*}{V} \right) \right] = nq_3 (1 - q_3) \left[\log \left(\frac{u_3}{d_3} \right) \right]^2 = \sigma_{pv} t$$

Further, for the means to be equal we have to have:

$$\left[q_3 \log \left(\frac{u_3}{d_3} \right) + \log d_3 \right] n = \mu_3 t$$

where σ_{pv} is the covariance and u_3 is the mean contributed by the third process, t is the time left for maturity of the bond and $3n$ is the total number of steps.

The alert reader will point out that there is no real equivalent of μ_3 . We set it equal to $\mu_v - r/3$.

Setting the values of the other parameters at (reasons for specific values for parameters can be clearly seen in appendix B)

$$q_3 = \frac{1}{2} + \frac{1}{2} \left(\frac{\mu_3}{\sigma_{pv} \sqrt{2}} \right) \sqrt{t/n}$$

and

$$u_3 = e^{+ \sqrt{\frac{\sigma_{pv} t}{n}}} \quad d_3 = e^{- \sqrt{\frac{\sigma_{pv} t}{n}}}$$

We get the covariance provided by the third step

$$\hat{\sigma}_{pv} n = \left[\sigma_{pv} - \mu_3 \left(\frac{t}{n} \right) \right] t$$

which in the limit tends to the required value

$$\hat{\sigma}_{pv} n \rightarrow \sigma_{pv} t \quad \text{as } n \rightarrow \infty$$

For the other two binomial processes, we have to have

$$n \{ q_1 (1 - q_1) \left[\log \left(\frac{u_1}{d_1} \right) \right]^2 + q_3 (1 - q_3) \left[\log \left(\frac{u_3}{d_3} \right) \right]^2 \}$$

$$= \sigma_p^2 t$$

Or in the limit as $n \rightarrow \infty$

$$n [q_1 (1 - q_1) (\log (\frac{u_1}{d_1}))^2] = [\sigma_p^2 - \sigma_{pv}] t$$

Similarly,

$$n [q_2 (1 - q_2) (\log (\frac{u_2}{d_2}))^2] = [\sigma_v^2 - \sigma_{pv}] t$$

For the means of the distributions to be equal we require that

$$\lim n \rightarrow \infty [q_1 \log (\frac{u_1}{d_1}) + q_3 \log (\frac{u_3}{d_3}) + \log d_1 d_3 \hat{r}] n = \mu_p t$$

and similarly for $\mu_v t$

We know that the discount rate per period \hat{r} should satisfy $\hat{r}^{3n} = e^{rt}$ where r is the annualized risk free rate and t the time to maturity in years.

By setting

$$u_1 = e^{+(\sqrt{\sigma_p^2 - \sigma_{pv}}) \sqrt{\frac{t}{n}}} \quad d_1 = e^{-(\sqrt{\sigma_p^2 - \sigma_{pv}}) \sqrt{\frac{t}{n}}}$$

$$q_1 = \frac{1}{2} \left[1 + \frac{(\mu_p - (\mu_3 + r/3)) \sqrt{\frac{t}{n}}}{\sqrt{(\sigma_p^2 - \sigma_{pv})}} \right]$$

$$u_2 = e^{\sqrt{(\sigma_v^2 - \sigma_{pv})} \sqrt{\frac{t}{n}}} \quad \cdot d_2 = e^{-\sqrt{(\sigma_v^2 - \sigma_{pv})} \sqrt{\frac{t}{n}}}$$

and

$$q_2 = \frac{1}{2} \left[1 + \frac{(\mu_v - (\mu_3 + r/3))}{\sqrt{(\sigma_v^2 - \sigma_{pv})}} \sqrt{\left(\frac{t}{n}\right)} \right]$$

we can show that the required values hold in the limit.

Note that μ_3 is arbitrary and q_1 and q_2 play no part in the valuation of the bond except to reassure us that the processes are identical. So far we have only assured ourselves that the means and the variances of the binomial process can be made to tend to the required values. We will show in Appendix B that the process tends in the limit to the same probability distribution as the bivariate normal.

IV. EXTENSIONS

- (1) Payouts by firm: If δ is the fraction of firm value paid out as dividend every year, it can be incorporated by diminishing the firm value every $\frac{n}{t}$ iterations by δV . Bankruptcy would not occur as the value of the firm could never go to zero as a result of a fractional payout.
- (2) Coupon payments on bond: If C is the yearly fixed coupon payment on the bond, it could be depicted by diminishing the value of the firm every $\frac{n}{t}$ periods by C (and checking for default). The net coupon (after default) could be added to the bond value at that node and the standard process could be followed to evaluate the bond value.
- (3) Stochastic interest rate: This could be incorporated by having a fourth step (plus more for covariance terms).
- (4) Senior debt: Senior debt could be incorporated by changing the terminal conditions:
i.e., if S be the amount of senior debt, the bond value at maturity
$$= \text{Min} [V - S, F + \max [0, P - E]]$$
- (5) Convenience yield on the commodity option can be treated in the same way as dividends on a stock option (see Fall 1986 for proof).
i.e., if C_1 is the convenience yield per period, it diminishes the value of the commodity price by $(1 - C_1)$ every period.
- (6) Terminal Conditions: Different terminal conditions could be incorporated by merely changing the function which describes the bond value on terminal date. Nothing else will have to change. Hence, an

Indexed Commodity Option Note, which has a sliding stream of payments on maturity date with the underlying amount itself being a function of the price, can easily be priced. Pricing a cap is a trivial extension.

V. COMPARATIVE ANALYSIS OF BINOMIAL MODEL AND SCHWARTZ MODEL RESULTS

The model described earlier was programmed using Turbo Basic on an IBM PC AT. First, the case assumed by Schwartz was used as a check. The extensions possible with this model were then incorporated and priced. Checks were made by taking extreme cases where we know the expected result.

The basic case assumed by Schwartz is that of a company having issued a zero coupon with face value $F=100$, maturing in five years. At maturity date the bondholder has the right to buy a certain commodity bundle with initial value P and price volatility σ_v and is correlated with the commodity price movement with correlation coefficient ρ .

Table 1 shows the price of the bond for various values of the covariance between the commodity price and the value of the firm as well as various values of the firm and the commodity bundle. The average difference in the prices obtained from the two models is about 0.3% with the maximum being 0.9% and the minimum being 0. This is after 10 iterations of the binomial model. In the limit the binomial model tends towards the Schwartz model. The advantage is not just simplicity, the binomial method enables us to incorporate senior debt, payouts by the firm before the maturity of the bond (in terms of coupons and dividends), interim coupons or options linked to the commodity bond and the risk of default thereof, stochastic interest rates, caps, etc. This can be done in a simple and intuitive manner; which is important because few bonds are identically structured and deriving the corresponding differential equation as well as solving it, even if computationally feasible, may be uneconomical. The following observations can be made from the results in Table 1.

Table 1: Commodity-Linked Bond Values for Different Commodity Bundle Prices, Firm Values and Correlations Using the Binomial Pricing Model

($E=F=100$, $R=0.12$, $T=5.0$, $\sigma_p=0.4$, $\sigma_v=0.3$, Number of Iterations $N=10$)

Firm Value, $V=200$					
		$P=0,0$	0,35	0,70	
$P=100$	Binomial	86,22	93,59	103,19	
	Schwartz	85,45	93,34	102,54	
	Difference (\$)	0,80	0,27	0,63	
$P=80$	Binomial	77,48	83,46	89,46	
	Schwartz	77,34	83,20	89,26	
	Difference (\$)	0,18	0,31	0,22	
$P=50$	Binomial	65,38	67,83	69,76	
	Schwartz	65,01	67,67	69,62	
	Difference (\$)	0,57	0,24	0,20	
Firm Value, $V=400$					
		$P=0,0$	0,35	0,70	
$P=100$	Binomial	99,89	105,00	109,16	
	Schwartz	99,00	104,66	108,70	
	Difference (\$)	0,90	0,32	0,42	
$P=80$	Binomial	86,88	90,53	92,30	
	Schwartz	86,57	90,02	92,35	
	Difference (\$)	0,36	0,57	-0,05	
$P=50$	Binomial	69,30	70,22	70,59	
	Schwartz	68,89	70,14	70,58	
	Difference (\$)	0,60	0,11	0,01	
Firm Value, $V=1000$					
		$P=0,0$	0,35	0,70	Def. Free
$P=100$	Binomial	107,95	109,08	109,58	109,41
	Schwartz	107,15	108,92	109,40	109,41
	Difference (\$)	0,75	0,15	0,16	0,00
$P=80$	Binomial	91,59	92,50	92,42	92,42
	Schwartz	91,45	92,41	92,60	92,60
	Difference (\$)	0,15	0,10	-0,20	-0,19
$P=50$	Binomial	70,64	70,58	70,61	70,61
	Schwartz	70,39	70,61	70,64	70,64
	Difference (\$)	0,36	-0,04	-0,04	-0,04

- (i) Effect of correlation between commodity bundle price and firm value: The greater the correlation, the less is the risk of default and hence the greater the bond value. The intuition is plain; if the correlation is higher, the chances are that when the bond payments are higher because of a high commodity price, the firm value will also be higher so that it will be able to repay without defaulting.
- (ii) Effect of firm value: The higher the firm value compared to the face value of the bond and the commodity bundle, the less the risk of default and hence the greater the bond value. However, as the firm value becomes very high compared to the potential bond obligations, the risk of default approaches an asymptotic limit--the default free bond value.
- (iii) Effect of higher commodity bundle price as compared to the exercise price: The greater this difference, the higher the bond value. However, if the price rises so high that the firm will default continuously, then the bond will assume a value approaching the expected value of the firm's assets.
- (iv) Effect of senior debt: The existence of senior debt diminishes the value of the bond as the default risk goes up. The higher the value of the firm, the lower the effect of senior debt on the bond value (see also Table 2). Also, the higher the correlation between firm value and commodity price, the less the effect of senior debt. Note that when we refer to senior debt we mean the senior debt which matures at the same time as the bond. Any debt maturing earlier is taken as a payout by the firm.

Table 2: Effect of Senior Debt (S) on Firm Value (V)

Firm Value (V)	V=200			V=400		
	S=0	S=100	Diff. (%)	S=0	S=100	Diff (%)
Re=0.35	93.59	76.25	18.50	105.00	100.65	4.14
Re=0.7	103.19	88.28	14.40	109.16	107.29	1.70

(v) Effect of cap: A cap is equivalent to a call option bought by the issuer from the buyer. Thus, the bond value should be diminished by the value of the call option with exercise price equal to the cap. But in Table 3 we see that the value is not diminished by the full value of the option. This is because the issuer would not pay for the high commodity price status in which he would declare bankruptcy. Therefore, an increase in the risk of default on the bond would decrease the value of a cap. In the limit, a cap would have no value if the bond always defaulted and paid nothing, while it would equal the value of the option if there was no default risk.

We now go beyond the Schwartz model and make the additions that are permitted by the binomial model. We will start with the basic bond and add features so that we can gain a sense of what each feature does to the price of the bond. We make the following assumptions:

Face value = $F = 100$	Time to maturity = 4 years
Exercise price = 100	Initial commodity price = 100
Coupon = 100	$\sigma_p = 0.4$
$\sigma_v = 0.3$	$P = 0.7$
Risk free rate = 0.12	

We will also assume no default risk initially.

Table 3: Effect of a Commodity Price Cap on the Commodity Bond Value

Firm Value (V)	With Cap=105	No Cap	
	Case A	Case B	Case C
V=200	66.97	93.59	26.62
V=400	68.32	105.00	36.68
V=1000	68.40	109.08	40.38

Note: Black-Scholes value of cap = 39.74.

We see from Table 4 that a coupon adds value to the bond and convenience yield diminishes the value of the bond. A cap, in the absence of default risk, reduces the value of the bond by the value of an option with exercise price equal to the cap (from the Black formula, the value of the cap is estimated at 21.28 as compared to the 21.60 we obtain). Increased payout and senior debt have no effect if we do not consider default risk. However, in the presence of default risk, senior debt diminishes the value of the bond and so do payouts to equity or other bonds. The cap, however, will be worth less.

Some Comparative Statics

The various parameters will be now varied for the above bond and the values of the zero-bond (principal and option repayment) and the coupons will be established.

Table 4: Impact of Additional Features on Bond Value

	Bond Value	Incremental Value
Original Bond	110.62	
<u>Additional Features</u>		
Coupon @ 10%	140.52	29.90
Convenience Yield @ 5%	124.94	-15.58
Cap at 150	103.34	-21.60
Default Risk ($v=200$)	100.70	-2.64
Senior Debt ($=50$)	93.46	-7.24
Payout Ratio (1.0) of Firm	80.45	-13.01

Effects of varying:

- (i) **Firm value**. Coupons are paid whenever they are due. Therefore, the default on the coupon is only likely when the firm value is comparable to the size of coupon payments. This can be seen in Figure 2 where default on the coupon starts only when the initial firm value is below 50. However, above 50, the coupon is not defaulted on and maintains a constant value. Similarly, default on the principal and option repayment becomes negligible at a firm value higher than 600.
- (ii) **Coupon rates**. Higher coupon rates increase the present value of the coupon but simultaneously decrease the value of the zero bond (Figure 3). This is because a higher coupon diminishes the value of the firm more and leaves a lower amount to repay the principal/option. The net effect is that a higher coupon does not increase the value of the bond as much in the presence of default risk as it would a default free bond.

BOND VALUES FOR DIFFERENT FIRM VALUES

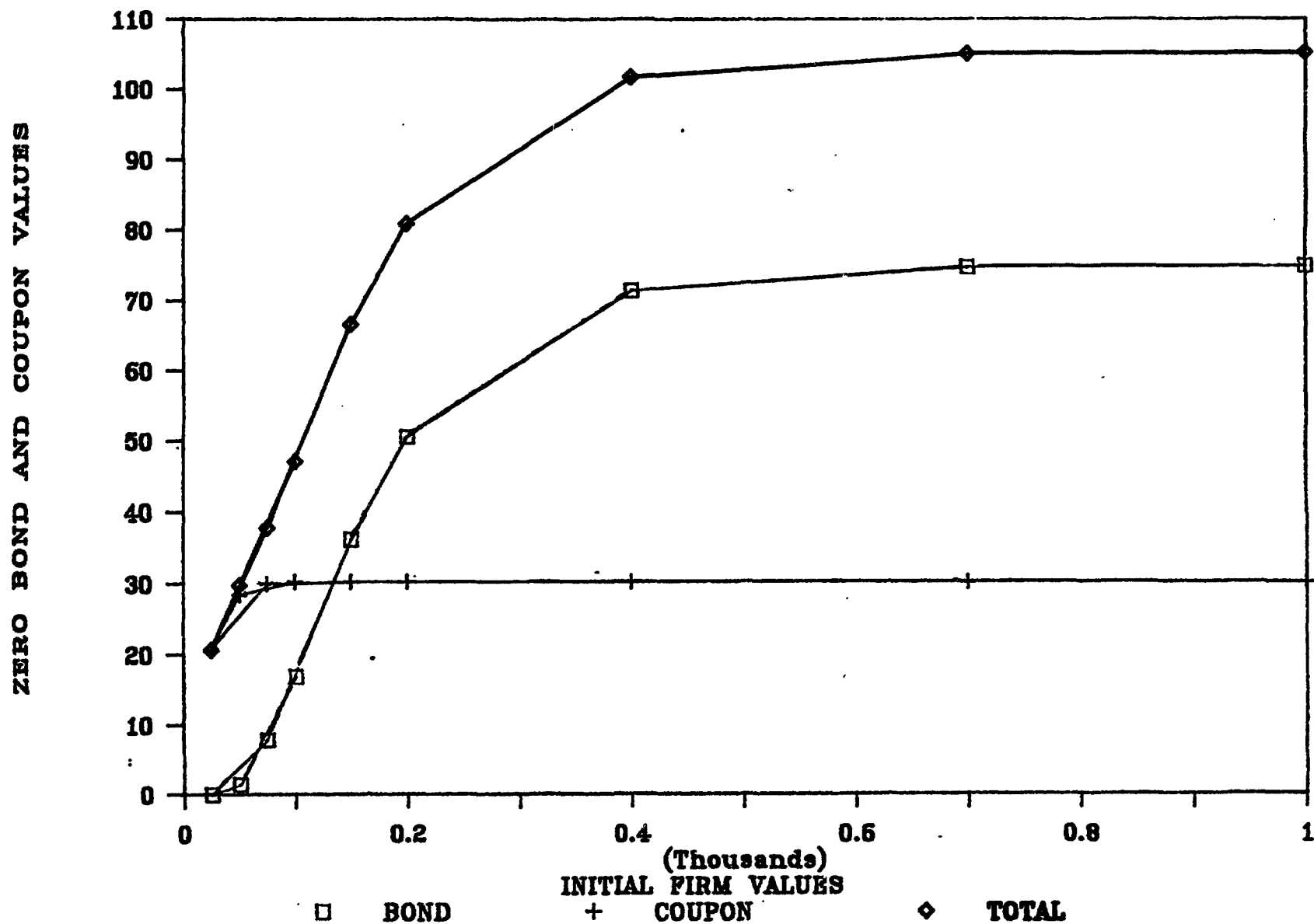
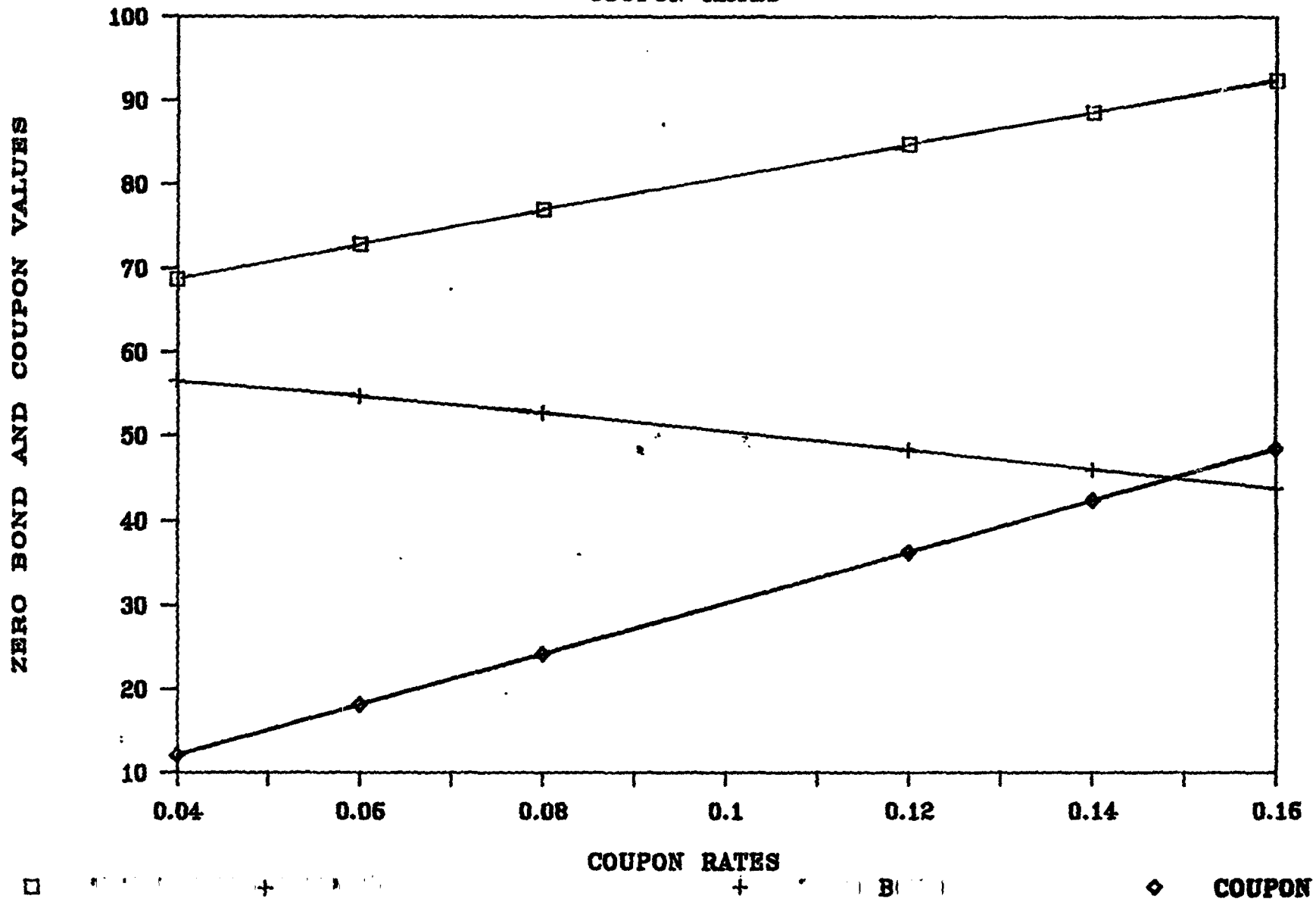


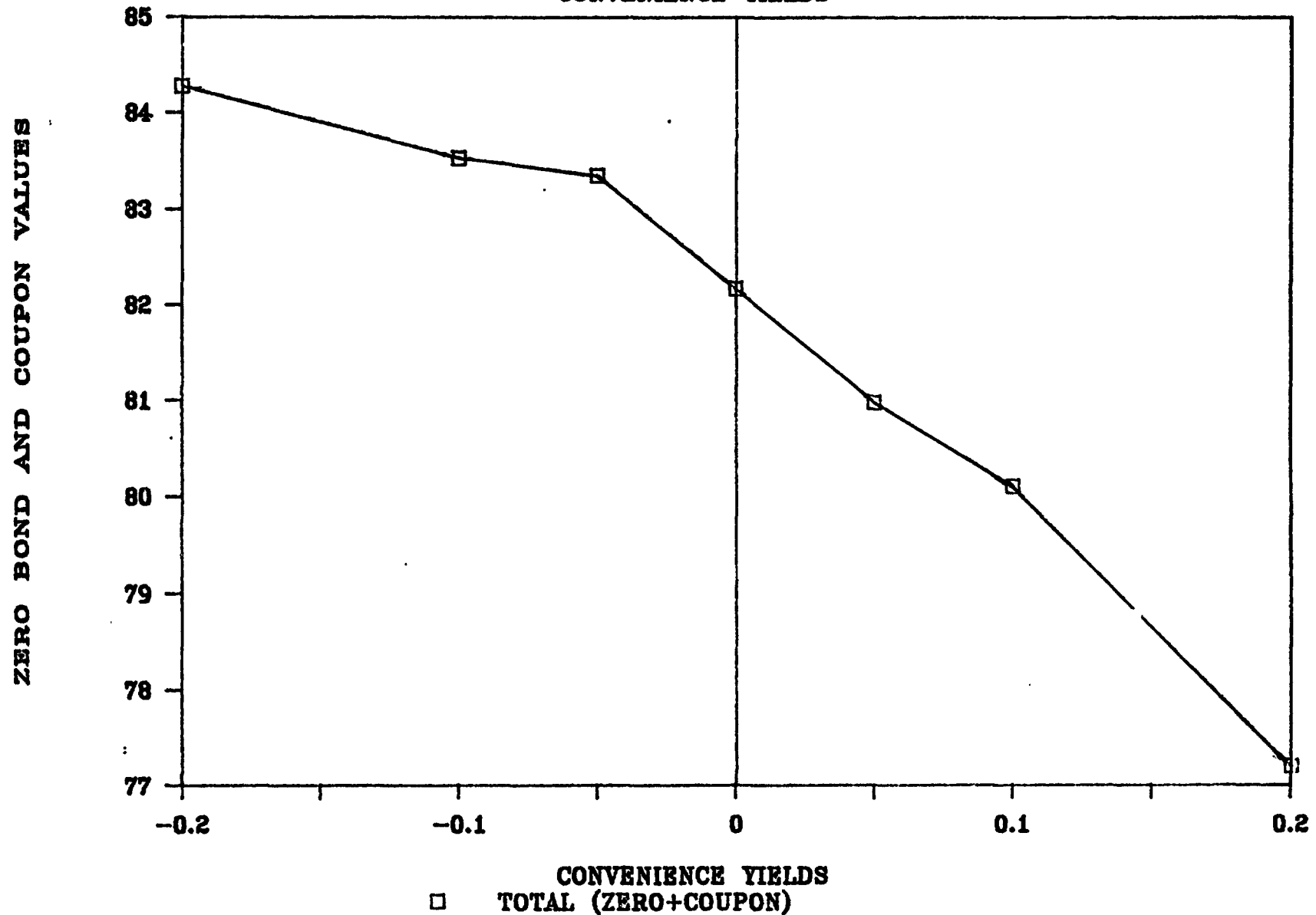
Figure 3

BOND VALUES FOR DIFFERENT COUPON RATES



- (iii) Convenience yields. The effect of convenience yields is important as these are fairly volatile for some commodities, like oil. From Figure 4 it can be seen that a sharp change in convenience yields, e.g., from 20% to -20%, changes the total bond value by about 10%.
- (iv) Senior debt. We refer here to debt maturing at the same time as the bond but being senior to the bond. The larger the senior debt, the greater the chance of default on the principal/option, as seen in Figure 5. Dividend or other payouts earlier than the bond maturity have a similar effect (Figure 6).
- (v) Caps. Caps are effective so long as they are at price levels which have high probabilities of being attained; at higher levels they are of negligible value (Figure 7).
- (vi) Correlation. Correlation between the firm value and the commodity bundle price decreases the default risk and hence the value of the bond (Figure 8).

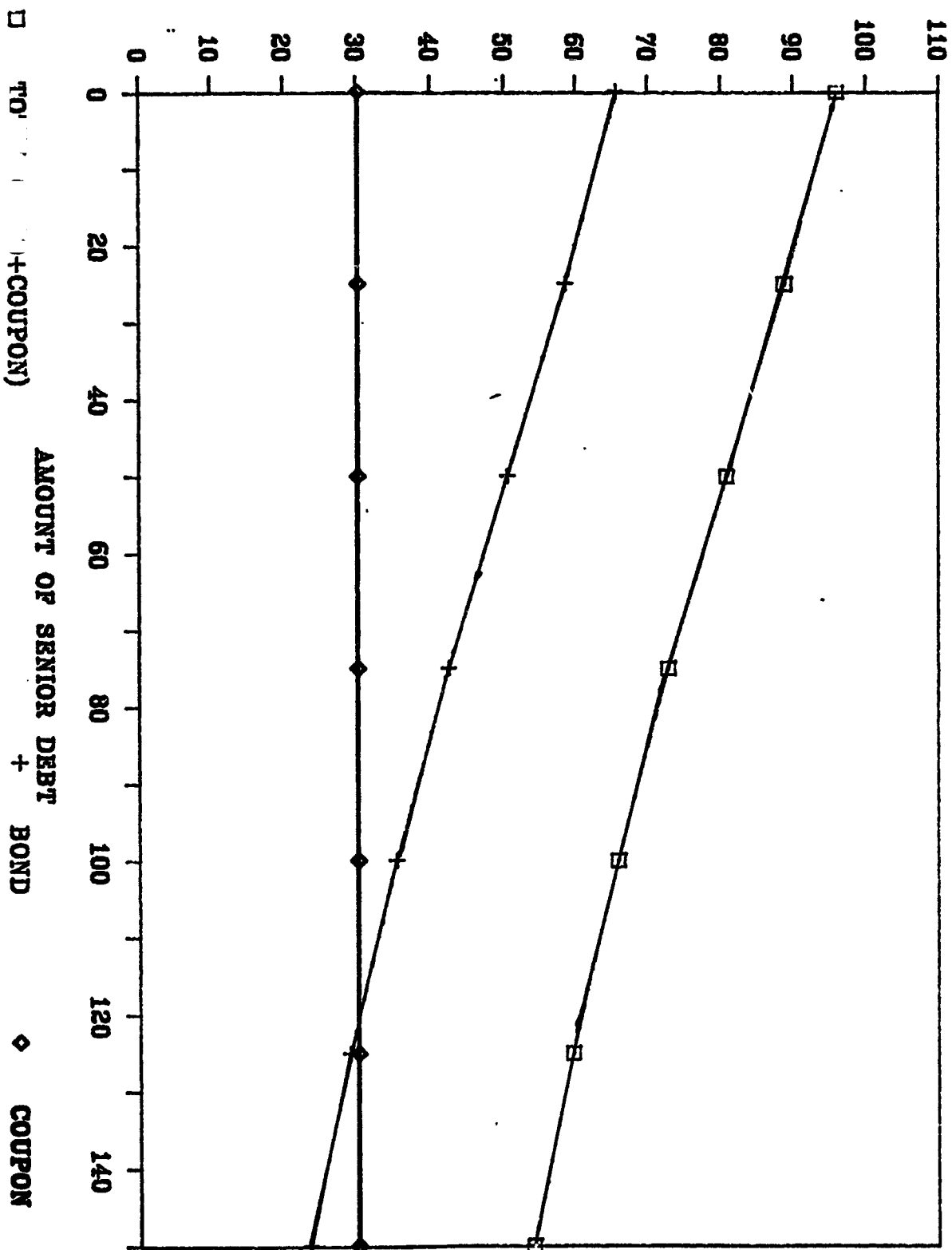
BOND VALUES FOR DIFFERENT
CONVENIENCE YIELDS



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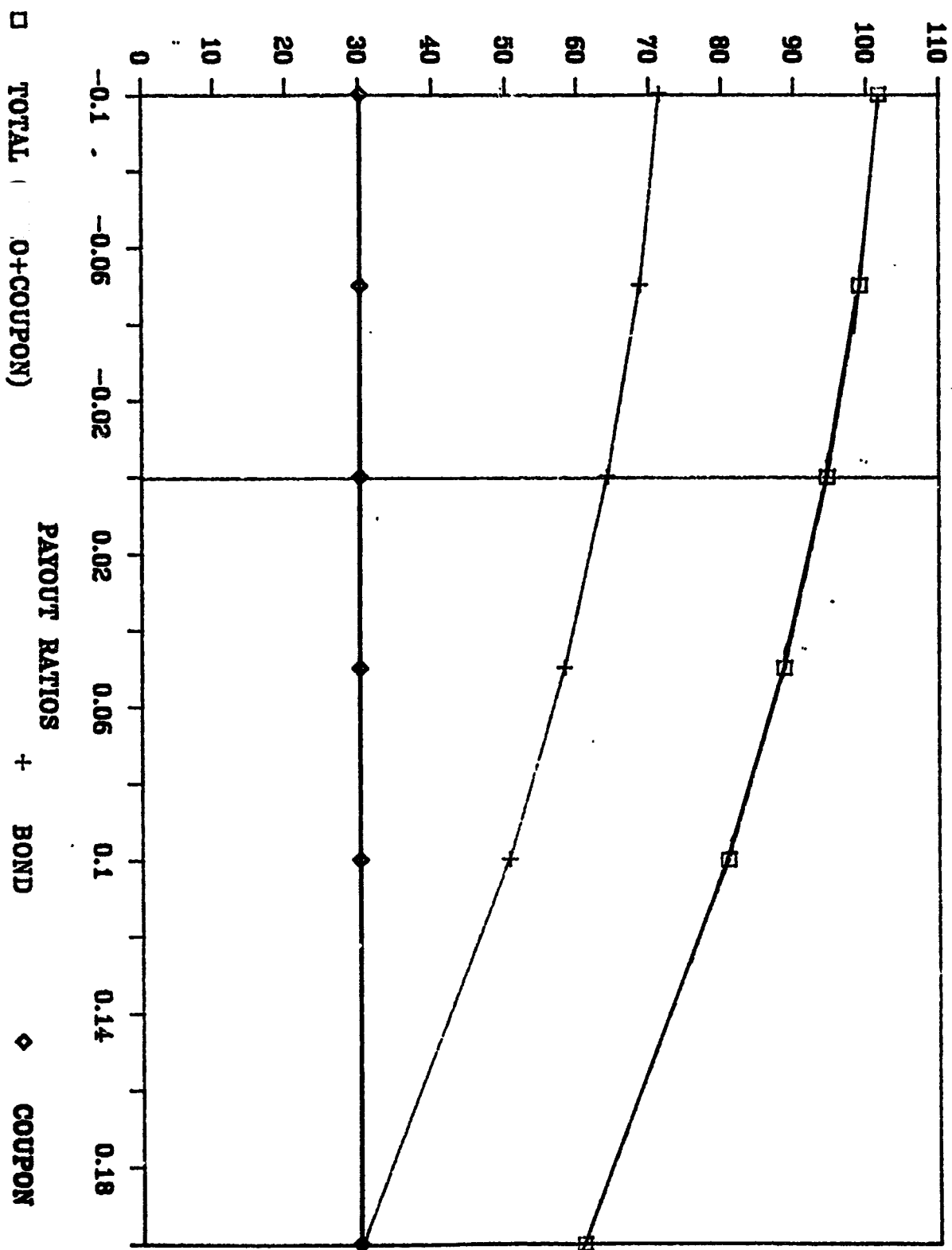
DEBT AMOUNTS

ZERO BOND AND COUPON VALUES

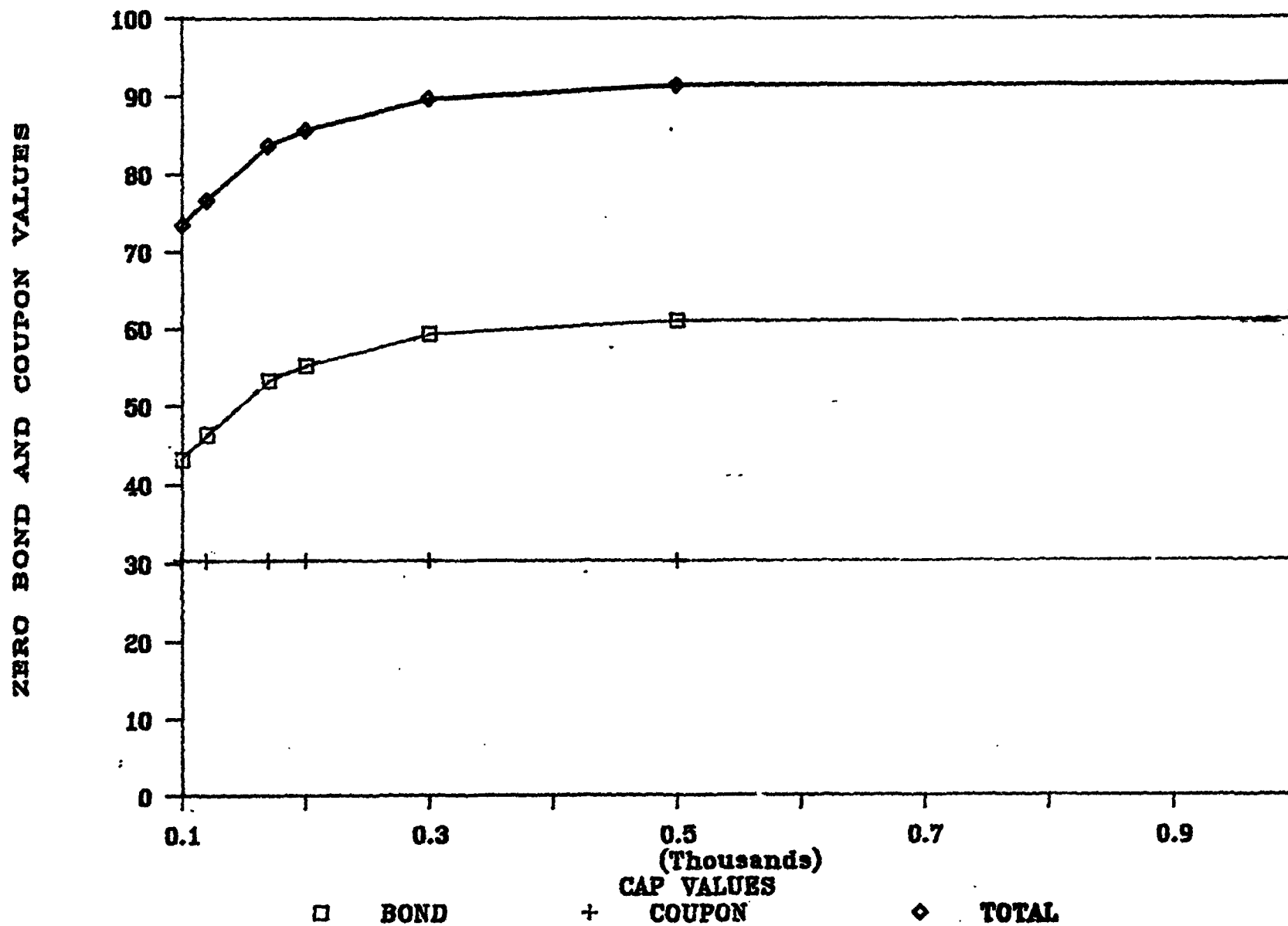


BOND VALUES FOR DIFFERENT PAYOUT RATIOS

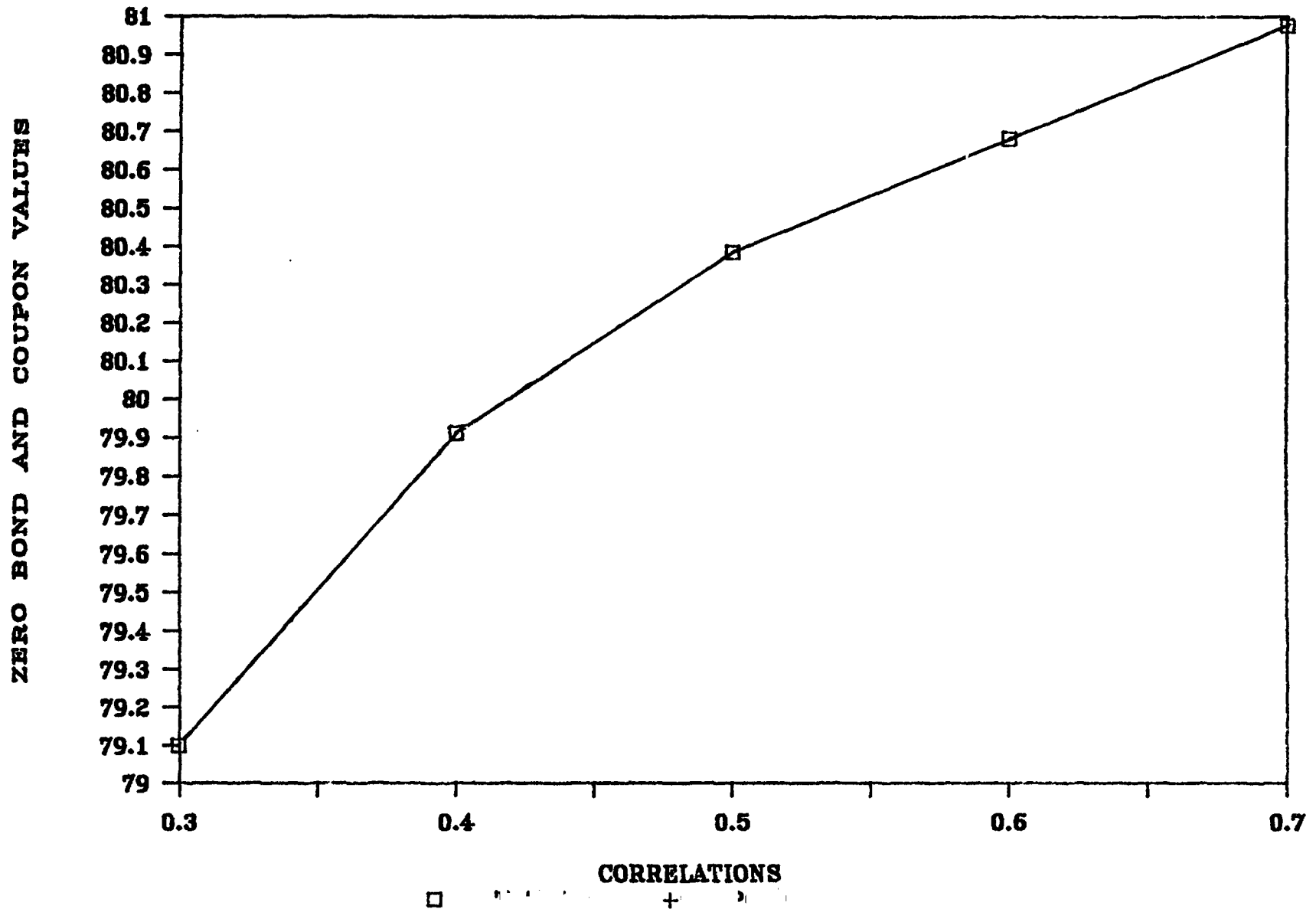
ZERO BOND AND COUPON VALUES



BOND VALUES FOR DIFFERENT CAPS



BOND VALUES FOR DIFFERENT CORRELATIONS



VI. CONCLUSION

The binomial model is an effective way of pricing a commodity bond in the presence of commodity price risk and default risk. Extensions to incorporate other sources of risk can be easily made. The limiting factor in all this is computational power, but it becomes significant only in the presence of features like fixed coupon payments or fixed payouts.

The application of this intuitive method to commodity-linked bonds is just one of the many applications possible. For example, secondary market developing country debt could be priced by suitably redefining V (the value of the firm) and P (the price of the commodity).

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APPENDIX A

THE CONTINUOUS TIME MODEL

Using the traditional continuous time option pricing method it can be shown that if the price of the commodity bundle P and the value of the firm V follow stochastic processes:

$$\frac{dP}{P} = \mu_p dt + \sigma_p dZ_p$$

$$\frac{dV}{V} = \mu_v dt + \sigma_v dZ_v$$

and

$$dZ_p dZ_v = \sigma_{pv} dt$$

the differential equation to be solved is

$$\begin{aligned} & \frac{1}{2} \sigma_p^2 P^2 B_{pp} + \frac{1}{2} \sigma_v^2 V^2 B_{vv} + \sigma_{pv} PV B_{pv} \\ & + PB_p (r - \delta) + B_v [rV - D] - B_z - rB + C = 0 \end{aligned}$$

B is the value of the bond.

Z is the time to expiration.

δ is the convenience yield.

D is the total payout by firm per year.

C is the yearly coupon attached to the bond.

The boundary conditions are:

$$B(P, V, 0) = \min [V, F + \max [0, P - E]]$$

where F is the face value of the bond.

If we assume the payout D is a fixed fraction d of the firm value V and the coupon C is a fixed fraction c of the face value of the bond, we have to check for default every time the coupon is paid.

$$\text{i.e., } V \geq C$$

The solution to this equation (if at all possible) would be very cumbersome even by most numerical methods.

APPENDIX B

PROOF THAT THE DISTRIBUTION OF THE BINOMIAL MODEL TENDS TO THE BIVARIATE NORMAL DISTRIBUTION

To show that the binomial model tends in the limit to the Bivariate Normal Distribution, we show that the characteristic function of the former tends towards the latter.

Let us consider the three step process in Figure 1. There are eight terminal nodes at D which we will number from top to bottom 1 to 8. At the top-most node, the commodity price is Pu_1u_3r . Therefore, the log of the return on the commodity over the three steps at node 1 is

$$\log R_1 = \log u_1 + \log u_3 + \log \hat{r}$$

Similarly, the log return on the firm is

$$\log R_2 = \log u_2 + \log u_3 + \log \hat{r}$$

We want to determine the characteristic function of joint returns

$(\log R_1, \log R_2)$ which we shall denote as $\phi(\theta_1, \theta_2)$

$$\phi(\theta_1, \theta_2) = E[\exp(i\theta_1 \log R_1, i\theta_2 \log R_2)]$$

The expectation over the three step process is the sum of eight terms, each arising from a particular outcome of $(\log R_1, \log R_2)$

$$\phi(\theta_1, \theta_2) = \sum_{i=1}^8 D_i$$

where

$$D_1 = q_1 q_2 q_3 \exp[i\theta_1(\log u_1 + \log u_3 + \log \hat{r}) + i\theta_2(\log u_2 + \log u_3 + \log \hat{r})]$$

$$= \frac{q_1 q_2}{2} \frac{[1 + \mu_3/\hbar]}{\sigma_3} \exp[i\theta_1(\sigma_3/\hbar + \sigma_a/\hbar + \log rt/3n) + i\theta_2(\sigma_3/\hbar + \sigma_b/\hbar + \log rt/3n)]$$

where $h=t/n$, $\sigma_3 = \sqrt{\sigma_{pv}}$, $\sigma_a = \sqrt{(\sigma_p^2 - \sigma_{pv})}$, $\sigma_b = \sqrt{(\sigma_v^2 - \sigma_{pv})}$

$$D_1 = \frac{q_1 q_2}{2} \frac{(1 + \mu_3 \sqrt{h})}{\sigma_3} \exp \{ \sqrt{h} [i\theta_1(\sigma_3 + \sigma_a) + i\theta_2(\sigma_3 + \sigma_b)] + h[i\theta_1 r/3 + i\theta_2 r/3] \}$$

Expanding the exponential as a power series, multiplying out and rearranging, we get

$$\begin{aligned} D_1 = & \frac{q_1 q_2}{2} \frac{(1 + \sqrt{h} [\frac{\mu_3}{\sigma_3} + i\theta_1(\sigma_3 + \sigma_a) + i\theta_2(\sigma_3 + \sigma_b)])}{\sigma_3} \\ & + h [\frac{\mu_3}{\sigma_3} i\theta_1(\sigma_3 + \sigma_a) + \frac{\mu_3}{\sigma_3} i\theta_2(\sigma_3 + \sigma_b) + i\theta_1 r/3 + i\theta_2 r/3 - \frac{\theta_1^2}{2}(\sigma_3 + \sigma_a)^2 - \\ & - \theta_1 \theta_2 (\sigma_3 + \sigma_a)(\sigma_3 + \sigma_b) - \frac{\theta_2^2}{2}(\sigma_3 + \sigma_b)^2] + o(h) \end{aligned}$$

where $o(h)$ indicates powers of h higher than 1 which will be negligible in the limit.

Summing over all the eight nodes (i.e., finding the corresponding expression to D_1 above one for D_2, D_8 and then adding them all together), a tedious but necessary process, and then simplifying, we get

$$\begin{aligned} \phi(\theta_1, \theta_2) = & 1 + \sqrt{h} [i\theta_1 \sigma_a (2q_1 - 1) + i\theta_2 \sigma_b (2q_2 - 1)] + \\ & + h [i\theta_1 \mu_3 + i\theta_2 \mu_3 + i\theta_1 r/3 + i\theta_2 r/3 + \frac{\theta_1^2}{2}(\sigma_3^2 + \sigma_a^2) + \frac{\theta_2^2}{2}(\sigma_3^2 + \sigma_b^2) + \theta_1 \theta_2 (\sigma_3^2 \\ & + (2q_1 - 1)(2q_2 - 1) \sigma_a \sigma_b)] + o(h) \end{aligned}$$

Setting $q_1 = \frac{1}{2} [1 + (\mu_p - (\mu_3 + r/3)) \sqrt{\frac{t}{n}}]$, $q_2 = \frac{1}{2}$ and $\mu_3 = \mu_v - r/3$

and substituting back for σ_a , σ_b , σ_3 we get

$$\phi(\theta_1, \theta_2) = 1 + h \{ [i\theta_1 \mu_1 + i\theta_2 \mu_2] - \frac{1}{2} [\theta_1^2 \sigma_p^2 + \theta_2^2 \sigma_v^2 + 2\theta_1 \theta_2 \sigma_{pv}] \} + o(h)$$

We know that after n such sequences,

$$\phi_n(\theta_1, \theta_2) = [\phi(\theta_1, \theta_2)]^n \text{ from the independence of successive processes.}$$

Therefore, allowing $n \rightarrow \infty$ such that $h=t/n \rightarrow 0$, we get

$$\lim_{n \rightarrow \infty} \phi_n(\theta_1, \theta_2) = 1 + t \{ i\theta_1 \mu_p + i\theta_2 \mu_v - \frac{1}{2} (\theta_1^2 \sigma_p^2 + \theta_2^2 \sigma_v^2 + 2\theta_1 \theta_2 \sigma_{pv}) \}$$

But the characteristic function for the joint lognormal diffusion process with parameters $\mu_p, \mu_v, \sigma_p, \sigma_v$ is

$$\psi(\theta_1, \theta_2) = 1 + (i\theta_1 \mu_1 t + i\theta_2 \mu_2 t) - \frac{1}{2} (\theta_1^2 \sigma_p^2 t + \theta_2^2 \sigma_v^2 t + 2\theta_1 \theta_2 \sigma_{pv} t)$$

which is what we have as the limit of the binomial.

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